Weak localization in a chaotic periodically driven anharmonic oscillator

N. Ben-Tal and N. Moiseyev
Department of Chemistry, Technion–Israel Institute of Technology, Haifa 32000, Israel

S. Fishman
Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel

F. Bensch and H. J. Korsch
Fachbereich Physik, Universität Kaiserslautern, D-6750 Kaiserslautern, Germany
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The classical and quantum dynamics of a periodically driven anharmonic oscillator is studied. For a monochromatic field it is demonstrated that quantum mechanics imposes only weak limitations on the classical chaotic spreading, which are destroyed by introducing weak random noise in the field amplitude or when the time-reversal symmetry of the system is broken.

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Our numerical studies of the dynamics of the continuously driven anharmonic oscillator show a behavior which is markedly different from many systems studied so far [1–6]. For this particular system it will be demonstrated that there is an appreciably long time scale where the quantal survival probability is twice the classical one. Its origin is similar to weak localization in disordered systems [4, 6]. From the arguments presented it is expected that this is a generic phenomenon in chaotic systems. It holds for times shorter than the asymptotic time scale, which is required to resolve the individual levels. On the latter time scale the survival probability is enhanced by a different mechanism [7]. The weak localization is a dynamical phase effect and is not reflected in static calculations such as that of the Husimi distribution functions [5]. The model studied in the present work is a periodically driven quartic oscillator defined by the Hamiltonian

$$H(t) = \frac{p^2}{2m} + bx^4 - f_0 x \cos(\Omega t)\,, \tag{1}$$

which is a special case of the classical Duffing oscillator and was shown to support a large *bounded* chaotic zone [8, 5]. As shown in Fig. 1 for the parameter values $m=1$, $b=1/4$, $f_0=1/2$, and $\Omega=1$, the iterates of a single classical trajectory fill a bounded chaotic region in the Poincaré section of phase space. Such a trajectory generates a classical density in coordinate space [9], which is shown in Fig. 1.

The quantum quasienergy states for Hamiltonian (1) can be clearly classified into localized and extended states, where the latter are related to the classically chaotic phase-space region [5]. This suggests a definition of an average quantum "chaotic" probability density at x given by

$$P(x) = \frac{1}{N_{\text{ex}}} \sum_{\alpha} \int \Gamma_\alpha(x,p)\, dp\,, \tag{2}$$

where $\Gamma_\alpha(x,p)$ is the Husimi distribution function of the αth quasienergy state and the sum runs over all N_{ex} extended states. The quasienergy states were obtained by propagating the time-evolution matrix $U(0,t)$ by the Adams-Moulton predictor-corrector method in the basis-set representation of 360 variational eigenfunctions of the zero-field Hamiltonian, where $f_0=0$. The quasienergy states are given by

$$\psi_\alpha(x,t) = e^{-i\omega_\alpha t}\,\phi_\alpha(x,t) \tag{3}$$

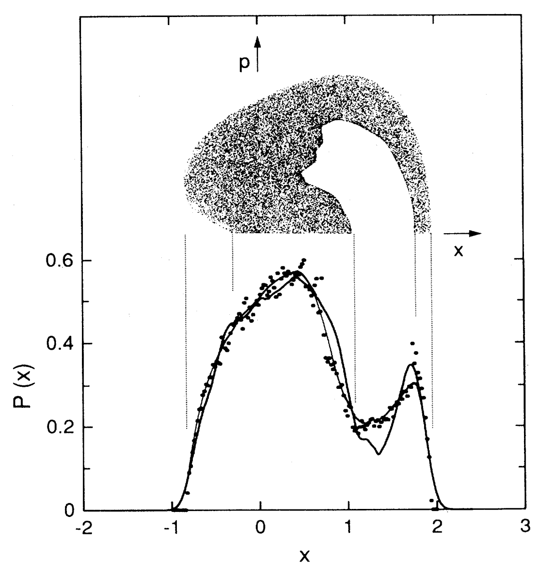

FIG. 1. The inset shows a Poincaré section of the classical chaotic region of phase space in the positive momentum half plane at $t=nT$ $(n=0,1,2,\ldots)$. All points result from a single classical trajectory. Quantum ($\hbar=0.015$) probability densities $P(x)$ (thick curve), defined in Eq. (2), are compared with the classical distribution computed from a single trajectory (dots). The thin curve is obtained by Gaussian smoothing of the classical histogram.

with $\phi_\alpha(x,t+T) = \phi_\alpha(x,t)$, where $\varepsilon_\alpha = \hbar\omega_\alpha$ are the quasienergies. The time-periodic wave functions $\phi_\alpha(x,0)=\phi_\alpha(x,nT)$ are associated with the eigenvalues $\lambda_\alpha = e^{-i\omega_\alpha T}$ and eigenfunctions of the time evolution matrix at $t=T$. It is easy to show that the quantum probability density $P(x)$ given by Eq. (2) is equal to the Gaussian average of the quasienergy solutions. The results presented in Fig. 1 show remarkable agreement between quantum ($\hbar = 0.015$) and classical probability densities and definitely demonstrate that the strong localization of the Husimi distribution in the classical chaotic phase space region (see Fig. 4 in Ref. [5]) is *not* a quantum phenomenon and does *not* provide an indication to quantum effects on classical chaotic dynamics.

In order to study classical versus quantum dynamics we compared the classical autocorrelation function $S_{\rm cl}(x,p,t)$ and the quantum survival probabilities $S_{\rm qm}(x,p,t)$ given by

$$S_{\rm cl}(x,p,t) = h\int\int \rho_0(x',p')\rho_t(x',p')\,dx'dp', \tag{4}$$

$$S_{\rm qm}(x,p,t) = \left|\langle\Phi(t)\,|\Phi_G\rangle\right|^2,$$

where $\Phi(x',t)$ is the time-evolved Gaussian wave function $\Phi_G(x') = \Phi(x',0)$ and $\rho_t(x',p')$ is the time-evolved classical phase-space density starting from an initial Gaussian ensemble $\rho_0(x',p')$, sampled by 10^3 normally distributed phase-space points with center (x,p) and width $\sigma = \sqrt{\hbar/2}$. The corresponding quantum-mechanical counterpart is a Gaussian minimum-uncertainty wave packet and the corresponding survival probability is

$$S_{\rm qm}(x,p,t=nT) = \left|\sum_\alpha \lambda_\alpha^n \left|\langle\Phi_G|\phi_\alpha(0)\rangle\right|^2\right|^2. \tag{5}$$

Note that $S_{\rm qm}$ can be precisely written in the form of $S_{\rm cl}$ in (5) when using the quantum Wigner phase-space densities.

In the absence of symmetries the time average of the quantum survival probability is

$$\overline{S}_{\rm qm}(x,p) = \sum_\alpha \left|\langle\Phi_G|\phi_\alpha(0)\rangle\right|^4. \tag{6}$$

The ratio between the quantum and classical survival probability is defined by

$$R(x,p,t) = \frac{S_{\rm qm}(x,p,t)}{S_{\rm cl}(x,p,t)}, \tag{7}$$

and similarly the ratio between the time-averaged survival probabilities is defined by $\overline{R}(x,p)$. To remove short-time fluctuations $S(x,p,t)$ is averaged over $2k+1$ periods leading to $\langle S\rangle_k(n) = \frac{1}{2k+1}\sum_{j=n-k}^{n+k} S(x,p,jT)$ and similarly $R_k(n)$ is defined as the ratio between $\langle S_{\rm qm}\rangle_k$ and $\langle S_{\rm cl}\rangle_k$. In our calculations $k=15$ is used. The smoothed quantum survival probabilities presented in Fig. 2(a) exhibit the suppression of the spreading generated by chaos. The long-time quantum localization for which $\overline{R}(0,0) \approx 2.8$ is due to quantum interferences. A possible explanation was given by Heller [7]. We shall come to this point later in the discussion of the dynamics in intermediate-time intervals. We argue that the dynamical quantum limitation is a pure phase effect. Indeed, by introducing 10% of random noise in the field amplitude, taking f_0 randomly distributed in the interval [0.45, 0.5], all phase interference effects are destroyed and a remarkable agreement between classical and quantum dynamics is observed [see Fig. 2(b)]. This could imply that under similar situations in experimental measurements the presence of random noise in the field intensity may provide a much closer agreement between quantum and classical dynamics than in the absence of noise. Neither weak nor strong quantum suppression of chaos may be observed. Note that in the random field calculations we could *not* make use of the Floquet theorem and $S(x,p,t)$ was obtained by direct time-dependent propagation rather than by Eq. (5).

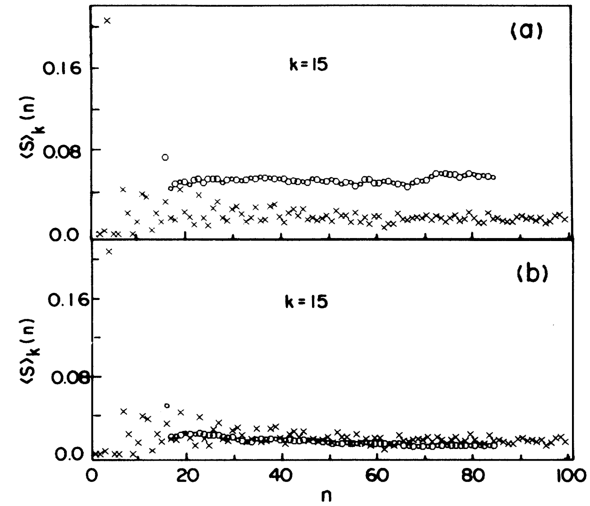

FIG. 2. Classical and quantum survival probabilities S at $t=nT$ ($T=2\pi$) obtained for an initial Gaussian distribution centered in the chaotic region at $x=p=0$. (a) The locally time averaged quantum survival probability $\langle S\rangle_k$ as a function of $t=nT$ shows an enhanced survival probability. The classical results are denoted by $\times$. (b) Same as (a) but under the influence of 10% random noise in the field amplitude.

For small values of $\hbar$ there is a regime where $S_{\rm qm}$ can be calculated semiclassically. The semiclassical propagator between two points (x',p') and (x'',p'') in phase space takes the form

$$K(x',p',x'',p'',t) = \sum_\gamma A_\gamma(x',p',x'',p'',t)\,e^{i\,W_\gamma(x',p',x'',p'',t)/\hbar}, \tag{8}$$

where the sum is taken over classical paths γ connecting these points in time t. The action W_γ is the integral of the Lagrangian along the path. The survival probability is $S_{\rm qm}(x,p,t) = \sum'{}'' \left|K(x',p',x'',p'',t)\right|^2$, where the sum $\sum'{}''$ is over classical paths with initial and final points (x',p') and (x'',p'') in the vicinity of (x,p). The size of this vicinity is that of the initial wave packet, namely of the order $\hbar$. It represents semiclassically the overlap of the initial wave packet with one that is evolved to time t. For chaotic systems with no symmetries the actions of different paths are unrelated and the interference term in the sum vanishes because of destructive interference, resulting from the pseudorandom phase factors. Consequently

$$S_{\rm qm}(x,p,t) = \sum{}' {}'' \sum_{\gamma} |A_\gamma(x',p',x'',p'',t)|^2 = S_{\rm cl}(x,p,t) . \quad (9)$$

These arguments do not hold if there are symmetries that relate various paths. For example, the Hamiltonian (1) satisfies time-reversal symmetry. Therefore, for each path starting in the vicinity of a point with zero momentum ($p = 0$) there is a time-reversed path that starts in this vicinity. The initial momenta of these two paths are opposite and therefore for this case each path corresponds to a time reversed path with identical values of A_γ and W_γ. Consequently for Hamiltonians with reversal symmetry [such as (1)] we have

$$S_{\rm qm}(x,p,t) = 2\sum{}' {}'' |A(x',p',x'',p'',t)|^2 = 2\, S_{\rm cl}(x,p,t) . \quad (10)$$

If the time-reversal symmetry of the Hamiltonian is broken, Eq. (9) also holds for $p=0$.

This argument, which is semiclassical in nature, breaks down after some time, because of the spreading of wave packets. This time cannot be longer than the time it takes to resolve the quasienergies. In what follows it will be demonstrated that there is an appreciable regime where these arguments hold. In particular it is found that time-reversal symmetry enhances localization. This enhancement is manifested by the fact that the quantal survival probability is larger than the classical one by a factor of 2. It is of the same origin as the weak localization effects in disordered solids [4]. Related effects were found for the kicked rotor [10] and for scattering systems [11].

In the absence of any symmetry in our model Hamiltonian (including time-reversal symmetry) the integral of the phase factor vanishes due to the strong "random" oscillations of W. In such a case one has, for an intermediate-time interval (50 cycles in the case studied here), *no* quantum enhancement of the classical chaotic survival probability and $R(x,p,t)\approx 1$, as shown in Fig. 3. After $t > 50\,T$ the discretization of the system is resolved and an oscillation is observed. The period of oscillation is approximately $100\,T$ and its inverse is of the order of the quasienergy spacing, which in turn is associated with the number of about 100 quantum states supported by the chaotic region (see Fig. 1). It should be stressed that the degree of quantum and classical correspondence for $t < 50\,T$ varies strongly, depending on the location of the initial Gaussian wave packet and the nature of the classical trajectories.

If, for example, it is centered on classical trajectories filling uniformly the entire bounded chaotic region at a time which is considerably smaller than $15\,T$, then the semiclassical arguments presented above do not hold. The ratio of $R(x,p,t) = 1$ is obtained if it takes more than 15 optical cycles for the classical Gaussian distribution to fill up uniformly the bounded chaotic region. The contributions from the cross terms $\gamma \neq \gamma'$ in calculating $S_{\rm qm}(x,p,t)$ from Eq. (8) vanish provided that there is *no* time-reversal symmetry. As discussed above,

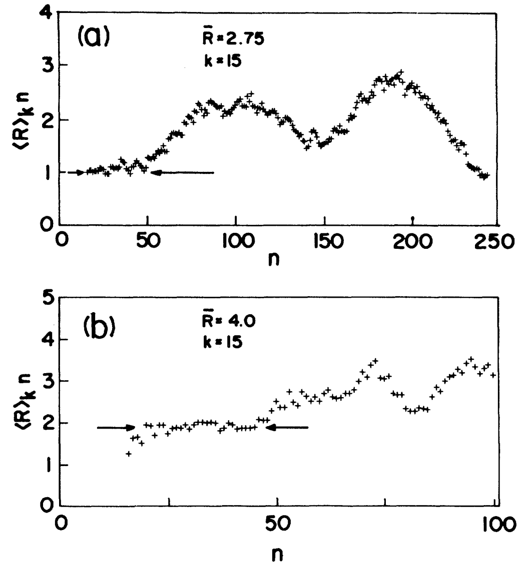

FIG. 3. The ratio between the short-time averaged quantum and classical survival probabilities $R_k(n)$ as function of time $t=nT$ is shown for two Gaussian distributions centered initially at (a) $(x,p)=(0.5,-0.3)$ and (b) $(0.75,0)$ in the classically chaotic region (see Fig. 1). $\overline{R}$ stands for the global time-averaged value and the arrows indicate the value of R obtained during the first 50 optical cycles before the discreteness of the quasienergy spectrum is resolved. (a) *No* quantal suppression ($R_k \approx 1$) of the chaotic dynamics is observed in contrast to (b), where the initial momentum is $p = 0$, yielding a ratio of $R_k \approx 2$.

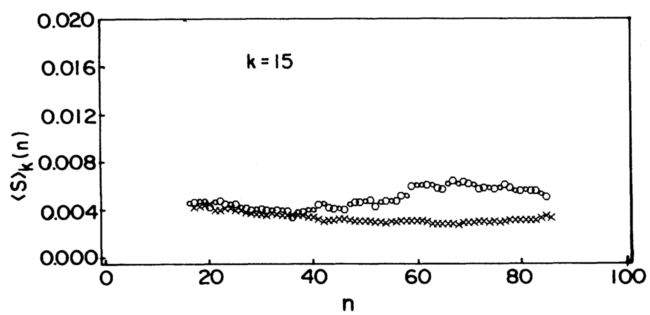

FIG. 4. Short-time time averaged quantum and classical survival probabilities obtained for the model Hamiltonian (11) with no time-reversal symmetry. The initial Gaussian distribution was centered at $x=p=0$.

when the initial wave packet is centered at $p=0$, we find $R(x,p=0,t)=2$ due to the time-reversal symmetry, whereas $R(x,p\neq 0,t)=1$.

The results presented in Fig. 3 confirm this analysis. During the first 50 optical cycles wave packets located initially in the chaotic region around $(x,p)=(0.75,0)$ lead to a value of $R_k=2$ [see Fig. 3(b)], whereas for a wave packet located at $p\neq 0$ [see Fig. 3(a)] the expected value of $R_k=1$ is obtained. The initial locations of the wave packet were selected on the basis of classical trajectory calculations showing that ensembles of Gaussian-distributed points centered at $(0.75,0)$ and $(0.5,-0.3)$ spread relatively slowly in time; thereby one would expect to avoid destruction of the wave packets during the intermediate-time scale.

The wave-mechanical enhancement in systems with time-reversal symmetry was discussed recently by Doron, Smilansky, and Frenkel [11], who suggested to destroy the time-reversal enhancement by introducing a strong enough magnetic field. Another possibility suggested here is to design a shape of the time-dependent field which is not time-reversal invariant, e.g.,

$$H(t)=\frac{p^2}{2}+\frac{x^4}{4}-\frac{x}{\sqrt{6}}\left(\cos t+\sqrt{3}\cos 2t+\sqrt{2}\sin t\right). \tag{11}$$

Indeed, the results presented in Fig. 4 for an initial Gaussian distribution centered at $x=p=0$ in the classical chaotic phase space region show that $R_k=1$ (i.e., $S_{\rm cl}=S_{\rm qm}$) during the first 40 optical cycles before the discreteness of the quasienergy spectrum is resolved.

Weak localization and its annihilation due to a random noise or by a specially shaped amplitude of the laser pulse can be experimentally observed in the rotational spectrum of diatomic molecules such as CsI under the influence of electromagnetic fields for which the chaotic phase space is bounded as in the periodically driven anharmonic oscillator discussed above.

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